Infinity Computing in Global and Local Optimization
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TARGETS OF THE RESEARCH
The research is dedicated to application of the Infinity Computer – a new type of supercomputer able to work numerically with infinitesimals and infinite numbers in global and local optimization with costly and noisy objective functions. Important industrial applications: solution to expensive and ill-conditioned optimization problems in image processing and noisy data fitting.

GLOBAL AND LOCAL OPTIMIZATION
EXPENSIVE GLOBAL OPTIMIZATION PROBLEMS
A general nonlinear regression problem
\[ f(x) = \sum_{i=1}^{T} (y_i - \eta_i(x))^2 \rightarrow \min, \quad \Omega \subseteq \mathbb{R}^n, \quad N = 4q, \]
where \( y_i, 1 \leq i \leq T, \) are real-valued observations corrupted by noise, \( \eta_i(x, t) = \sum_{j=1}^{m} a \exp(d_j(t)) \sin(2\pi \omega_j t + \phi_j), \quad 1 \leq j \leq T, \quad (4q \) parameters to identify, see [3]).

A challenging problem: given a limited computational budget, it is required to find a good approximation of the global solution to a multiparametric and multimodal costly objective function subject to nonlinear constraints.

A promising approach: extension of univariate methods to the multivariable case by means of diagonal space-filling curves ([8]).

APPLICATIONS IN NOISY DATA FITTING

METAOHEURISTIC VS DETERMINISTIC METHODS
Metahuristic (as genetic or other nature-inspired algorithms) are often used to study expensive black-box optimization problems.

However, the proposed deterministic methods (e.g., based on adaptive diagonal curves, ADC) demonstrate a much better performance with respect to widely used deterministic (e.g., DIRECT) and metaheuristic (e.g., genetic algorithm, GA) methods (see [4]).

INFINITY COMPUTING

AN ASTONISHING ANALOGY
Numerical system of the amazonian Pirahna tribe. They can count only 1, 2, many.

Many + 1 = many, many + 2 = many, many + many = many.

Traditional views on infinity:
\[ \infty + 1 = \infty, \quad \infty + 2 = \infty, \quad \infty + \infty = \infty. \]

NUMERICAL DIFFERENTIATION
Suppose that we have a computer procedure \( f(x) \) implementing the function \( g(x) = \frac{1}{x^2} \) and we want to evaluate the value \( f'(y) \) at the point \( y = 3 \). Numerical approximations are used for this purpose on traditional computers:
\[
\begin{align*}
  f'(x) &= \frac{f(x+h) - f(x)}{h} \\
  f'(y) &= \frac{f(y+h) - f(y)}{h} \\
  f'(3) &= \frac{f(3+h) - f(3)}{h}
\end{align*}
\]

GROSSONE
Grossone (\( \mathbb{G} \)) is the number of elements of the set of natural numbers. The principles of work with \( \mathbb{G} \) are the same as with finite numbers (see Ya. Sergeyev. Arithmetic of Infinity, CS, 2nd ed 2013):
\[
0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3...
\]

The non-contrastory nature of the methodology of grossone has been proven in [5]. Numerical system allowing one to execute operations with finite, infinite and infinitesimal numbers in a unique framework has been implemented on the Infinity Computer (see the patents [5]).

APPLICATIONS

Global and local optimization

Numerical differentiation

Ordinary differential equations

Tuning machines

Cellular automata

Sat system

Mathematical analysis

Hyperbolic geometry and tiling

Fractals and percolation, etc. (for details, see references in [7]).

INFINITY COMPUTING IN OPTIMIZATION

TRADITIONAL COMPUTERS — ERRORS

Underflows and overflows in traditional systems \( \rightarrow \) wrong solutions:

\[
\begin{align*}
 f'(x) &= \frac{f(x+h) - f(x)}{h} \\
 f'(y) &= \frac{f(y+h) - f(y)}{h} \\
 f'(3) &= \frac{f(3+h) - f(3)}{h}
\end{align*}
\]

INFFINITY COMPUTER — NO ERRORS
The Infinity Computer executes numerically the operations
\[
f'(3) = 2, \quad f'(3) = 0.5, \quad f'(3) = 2 - 0.25 = 0.5, \quad f'(3) = 3! - 0.125 = 0.75;
\]
being exact values of \( f(x) \) and the derivatives at the point \( y = 3 \).

TRADITIONAL COMPUTERS: ILL-CONDITIONING

Underflows and overflows in traditional systems \( \rightarrow \) wrong solutions:

\[
\begin{align*}
 f'(x) &= \frac{f(x+h) - f(x)}{h} \\
 f'(y) &= \frac{f(y+h) - f(y)}{h} \\
 f'(3) &= \frac{f(3+h) - f(3)}{h}
\end{align*}
\]

INFFINITY COMPUTER: WELL-CONDITIONING

\[
\begin{align*}
 f'(x) &= \frac{f(x+h) - f(x)}{h} \\
 f'(y) &= \frac{f(y+h) - f(y)}{h} \\
 f'(3) &= \frac{f(3+h) - f(3)}{h}
\end{align*}
\]

CONSTRAINED OPTIMIZATION: EXACT PENALTY

\[
\begin{align*}
 \min \quad & \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \\
 \text{subject to} \quad & x_1 + x_2 = 1
\end{align*}
\]

Penalty approach:
\[
\begin{align*}
 \min \quad & \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + P(1 - x_1 - x_2)^2 \\
 \text{subject to} \quad & x_1 + x_2 = 1
\end{align*}
\]

Traditional computers – iterative procedures with different \( P \) can return approximated solutions only.

Infinity Computer – exact penalty \( P \) (see [1]):
\[
\begin{align*}
 x_1^* &= \frac{1 - \sqrt{1 + \frac{3}{8} \left( 1 \frac{3}{8} \right)}}{2} \\
 x_2^* &= \frac{1 - \sqrt{1 + \frac{3}{8} \left( 1 \frac{3}{8} \right)}}{2}
\end{align*}
\]

The finite parts of \( x_1^* \) and \( x_2^* \) give us the exact solution to the original constrained problem: \( x = (\frac{1}{2}, \frac{1}{2}) \).

OBTAINED RESULTS
Infinity Computing has been successfully applied for solving important instances of ill-conditioned optimization problems [2,3]. New powerful multivariable optimization schemes have been proposed [4,6,9,10]: global optimization algorithms based on adaptive diagonal curves, acceleration techniques in derivative-free and smooth global optimization, grossone-based penalty functions in constrained optimization.

REFERENCES